

‘Plausibilities of plausibilities’: an approach through circumstances

Being part I of

“From ‘plausibilities of plausibilities’ to state-assignment methods”

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Abstract

Probability-like parameters appearing in some statistical models, and their prior distributions, are reinterpreted through the notion of ‘circumstance’, a term which stands for any piece of knowledge that is useful in assigning a probability and that satisfies some additional logical properties. The idea, which can be traced to Laplace and Jaynes, is that the usual inferential reasonings about the probability-like parameters of a statistical model can be conceived as reasonings about equivalence classes of ‘circumstances’ — viz., real or hypothetical pieces of knowledge, like e.g. physical hypotheses, that are useful in assigning a probability and satisfy some additional logical properties — that are *uniquely indexed* by the probability distributions they lead to.

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*If you can't join 'em,
join 'em together.*

0 Introduction

In the present first study we offer an alternative point of view on, or a re-interpretation of, probability-like parameters and ‘probabilities of probabilities’, two objects that appear in connexion with statistical models. This also provides a re-interpretation of some kinds of inverse methods, for which we develop a simple and general logical framework. This point of view, which we think is basically Laplace’s but uses an

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idea presented *in nuce* in some work by Jaynes, is alternative to both that based on the distinction between ‘physical’ and ‘subjective’ probabilities, and that based on de Finetti’s theorem. This point of view and the ensuing inverse-method framework have applications in physical theories, as will be shown in following studies [1, 2].

Our study and results are based exclusively on probability theory; we do not use entropy notions, for example. For us, ‘probability’ means simply *plausibility*, and the following conceptual proportion holds:¹

Plausibility calculus : Everyday notion of ‘plausibility’ =

Logical calculus : Everyday notion of ‘truth’.

We thus take the licence to adopt the term *plausibility* henceforth² — with no need to define what it means, any more than it is usually done with ‘truth’.³ (Our study can in any case be easily ‘translated’ into degrees-of-belief, credence, or similar terms.)

The notation $P(A|I)$ [5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] (cf. also [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]) will denote the plausibility of the statement A in the *context* described by the proposition I .⁴ We shall also say that I ‘leads to’ or ‘yields’ a given plausibility of A , but no particular meaning is intended with these two verbs. Associated plausibility densities will be denoted by $x \mapsto p_A(x|I)$; the term ‘distribution’ will be used for ‘density’ sometimes. Other symbols and notations are used in accordance with ISO [44] and ANSI/IEEE [45] standards.

In another study [46] we analyse the question of assigning plausibilities to unknown ‘events’ (e.g., measurement outcomes) from knowledge of ‘similar events’; a

¹Does also Wittgenstein mean something of the kind when he writes ‘Probability theory is only concerned with the state of expectation in the sense in which logic is with thinking’ [3, § 237][4, p. 231]? See also Johnson [5, pp. 2–3].

²Also used by Kordig [6].

³Is truth objective or subjective? Can truth be ‘operationally’ defined? Can the truth of a proposition be tested? — ‘Of course! to test the truth of ‘This hat is brown’ I only need to look at the hat!’ Well, provided e.g. that you are not dreaming or having hallucinations; and how do you test *that*? Going backwards, in the end you arrive at some proposition which you simply *assume* — unconsciously, by convention, by agreement, by caprice — and cannot ‘test’. ‘Subjectivity’ lurks no less in logic than in plausibility theory — and is no less *uninteresting* in plausibility theory than it is in logic.

⁴The context could also be called ‘condition’ or ‘situation’; Johnson calls it ‘supposal’ [5]; Jeffreys simply ‘data’ [17]. In the notation above, there is a relation (in the sense in which there is a relation between ‘certain’ and ‘true’) between the expressions ‘ $P(A|I) = 1$ ’ and ‘ $I \models A$ ’ (especially when this is used as in situational logic; see e.g. [38, 39, 40, 41]). The latter could also be suggestively written ‘ $T(A|I) = 1$ ’. The differences between the formalisms of logic and plausibility theory lie essentially in the fact that our everyday use of truth can be effectively (but not exclusively) *modelled* through a dichotomic set like $\{0, 1\}$ (or $\{\top, \perp\}$ or $\{T, F\}$), whereas our everyday use of plausibility can be effectively *modelled* through an ordered continuum like $[0, 1]$ (or $[0, +\infty]$ or $[-\infty, +\infty]$, see e.g. Tribus [42, pp. 26–29], Jaynes [20, ch. 2], Cox [12]). This has important implications, like the fact that a purely syntactic approach to plausibility, in the guise of the logical calculus, is near to worthless [43]. The parallel between plausibility theory and truth-functional logic suggests also another point. We do not require of logic, when put to practical use, that it should also provide us with the initial truth-value assignments (the ‘assumptions’). Why should we have an analogous requirement on plausibility theory with regard to initial plausibility-value assignments instead?

problem which is connected to induction. The key point is the formalisation, within probability theory, of the notion of ‘similar event’. This we do through the framework and the interpretation presented in the first note. We do not use the idea of exchangeability — and *infinite* exchangeability in particular — which is used in Bayesian theory for the same purpose; but there are known strong connexions and analogies with its mathematics and some of its results. In fact, we try to persuade the reader that our approach touches the core idea from which exchangeability also springs.

In a third note [1, 2] it will be shown that the inferential point of view presented here and in [46] finds applications in physical theories, like classical and quantum mechanics, an example being state reconstruction [47, 48]. The framework presented subsumes and re-interprets known techniques of quantum-state assignment (or ‘retrodiction’ or ‘reconstruction’) and tomography, and offers alternative approaches to analogous techniques in classical mechanics.

1 Statistical models and ‘probabilities of probabilities’

A statistical model is, roughly speaking, a plausibility distribution whose numerical values depend on parameters (for a critical discussion of more rigorous or useful definitions see [49, 50]). An example is the ubiquitous normal distribution

$$x \mapsto N(x | \mu, 1/\sigma^2) := \frac{1}{\sqrt{2\pi} \sigma} \exp\left[-\frac{(x - \mu)^2}{2\sigma^2}\right]$$

whose parameters are the expectation μ and the variance σ . Another example, one in which we shall be especially interested in this paper, is the ‘generalised Bernoulli’ model $i \mapsto \text{Br}(i | \mathbf{q})$, which gives the plausibility distribution for a set of m mutually exclusive and exhaustive propositions $\{R_1, \dots, R_m\}$, hereafter called *outcomes*,

$$p(R_i | \mathbf{q}) = \text{Br}(i | \mathbf{q}) := q_i, \quad (1)$$

depending on a set of parameters $\mathbf{q} := (q_1, \dots, q_m)$ which belong to a simplex of appropriate dimensionality:

$$\mathbf{q} \in \Delta := \{(x_i) \mid x_i \geq 0, \sum_i x_i = 1\}. \quad (2)$$

(This model is apparently called ‘discrete model’. Since this name is too anonymous and the model reduces to the Bernoulli one for $m = 2$, we opted for ‘generalised Bernoulli’ instead.)

The parameters of a statistical model are sometimes regarded as ‘unknown’, and a plausibility distribution (more precisely, a density) for them is therefore introduced. This distribution, usually called ‘prior distribution’ or simply ‘prior’, is used in calculations for a variety of purposes; two in particular interest us here and in the following papers. (1) A parameter-free plausibility distribution for the outcomes can be obtained integrating the product of the prior and the statistical model in respect of the parameters, i.e., by marginalisation. In the case of the Bernoulli model, e.g.,

introducing a prior $\mathbf{q} \mapsto f(\mathbf{q})$ (f being of course a normalised positive generalised function) one obtains the parameter-free distribution

$$p(R_i) = \int_{\Delta} \text{Br}(i|\mathbf{q}) f(\mathbf{q}) d\mathbf{q} \equiv \int_{\Delta} q_i f(\mathbf{q}) d\mathbf{q}. \quad (3)$$

(2) In so called ‘inverse methods’ or ‘problems’ (cf. Dale [51, §§ 1.2, 1.3]), the prior is used in the formula of Bayes’ theorem to obtain an ‘updated’ plausibility distribution for the parameters, conditional on knowledge of some outcome. The resulting distribution is called ‘posterior (distribution)’. In the case of the Bernoulli model, e.g., from the prior $\mathbf{q} \mapsto f(\mathbf{q})$ and knowledge of the outcome R_i one obtains the posterior

$$\mathbf{q} \mapsto f(\mathbf{q}|R_i) = \frac{p(R_i|\mathbf{q}) f(\mathbf{q})}{\int_{\Delta} p(R_i|\mathbf{q}') f(\mathbf{q}') d\mathbf{q}'}. \quad (4)$$

Such practices are at least as old as Bayes [52, 53, 54]. Related and unrelated historical information can be found e.g. in Dale’s book [51] and some nice essays by Hacking [55, 56, 57]; see also Jaynes’ discussion [20, ch. 18]. Old, though apparently not as old as Bayes, is also the question: how to interpret statistical-model parameters like \mathbf{q} and their prior distributions? The problem is that the parameters (q_i) look like plausibilities, since their values are identical to the plausibilities of the outcomes $\{R_i\}$ as eq. (1) shows, and that the prior f looks therefore like a ‘plausibility (distribution) of a plausibility’ — a redundant notion. This question, combined with the related issues on the interpretation of ‘probability’, has led to many philosophical debates; see e.g., amongst the vast literature on this, [58, 59]. The importance of the interpretative question is not merely philosophical, however. Different interpretations can lead to different conceptual and mathematical approaches — and thus to different solutions — in the investigation of concrete problems. This is particularly true for elaborate statistical models, like those connected to physical theories.

Two main interpretations appear to be in vogue. Many statisticians, logicians, and physicists, on the one hand, speak about ‘subjective’ and ‘physical’ probabilities (or ‘propensities’ [60]). For them the notion of a ‘probability of a probability’ poses no problems, since it means something like ‘the subjective probability of a propensity’. The very idea of ‘estimating a probability’ implies such kinds of interpretation; cf. e.g. Good [61], especially the title and chapter 2, Jamison [62], or Tintner [63].

For pious Bayesian or ‘de Finettian’ devotees, on the other hand, which conceive probability as ‘degree of belief’, the notion of a ‘degree of belief in a degree of belief’ is redundant or even meaningless. The Bayesian are notoriously rescued from philosophical headaches by de Finetti’s celebrated theorem and other similar ones [64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78], by which parameters like \mathbf{q} and functions like $\mathbf{q} \mapsto f(\mathbf{q})$ are introduced as mere *mathematical devices* — i.e., not plausibilities or degrees of belief! — that need not be directly interpreted. See Bernardo and Smith [76, ch. 4] for a neat presentation of this point of view. Interpretative issues like the Bayesian’s are also shared by those who thinks in terms of ‘logical probabilities’ [79] or, like we, simply in terms of ‘plausibilities’.

Here we present, discuss, and formalise still another interpretation — let us call it the ‘circumstance interpretation’ for definiteness’ sake, for reasons that will be apparent in the next section — in which functions like $q \mapsto f(q)$ do represent plausibility distributions, i.e., they are not mere mathematical devices, but the notion of ‘plausibility of a plausibility’ is nevertheless completely avoided. This interpretation combines two ideas by which Jaynes tried to make sense of ‘plausibility-like’ parameters: one is very briefly formulated in [80, p. 11], and can possibly be read also in de Finetti [81, § 20]; the other — the idea of an ‘ A_p distribution’ — appears in the various versions of his book on probability theory [18, lect. 18][19, lect. 5][20, ch. 18]. A similar interpretation is also proposed and discussed by Mosleh and Bier [59]. Caves also discusses, and criticises, a similar idea [78, 82]. It really seems to us, however, that this interpretation is basically what Laplace had in mind [53], if we read his ‘*causes*’ more generally as ‘*circumstances*’.

Instead of trying to summarise this interpretation in abstract general terms that would very likely only appear obscure at this point, we prefer to invite the reader to proceed to the simple and concrete example of the next section, just a coin toss away. The example will allow us to introduce the basic idea, along with some terminology. Then another, more elaborate example (§ 3) follows, to further expand the main idea. This is then abstracted and generalised (§ 4). Some important remarks are scattered throughout this note.

2 Interpreting plausibility-like parameters as ‘indexed circumstances’: introductory example

2.1 Context and circumstances

A coin has been tossed, the outcome unknown to us. We want to assign plausibilities to the outcomes ‘head’, R_h , and ‘tail’, R_t . The old recipe says to compute “*le rapport du nombre des cas favorables à celui de tous les cas possibles*” [54]. This is seldom of much help: Which are *the* cases? at which depth should the situation be analysed? And what if these cases are not equally plausible?

But why not analyse the situation in terms of some set of ‘cases’ anyway? *Some* set, not *the* set. And their plausibilities can be assigned by some other means. We do not want the ultimate analysis, just *an* analysis.

In our case, suppose that the knowledge of the situation, which constitutes the context I_{co} , says that either Cecily or Gwendolen or Jack or Algernon tossed the coin. Let us call these the four possible *circumstances* of the coin toss and denote them by C_C, C_G, C_J, C_A . The context could thus be analysed as the conjunction $I_{co} = J_{co} \wedge (C_C \vee C_G \vee C_J \vee C_A)$, for some ‘sub-context’ J_{co} .

Each circumstance says also something more about the respective person, which helps us in assigning the conditional plausibilities:⁵

⁵Cf. Laplace’s *Problème II* [53].

C_C : Cecily is a magician and skilled coin-tosser that always like to produce the outcome ‘head’. If we knew that she had tossed the coin, we would assign the distribution of plausibility

$$(P(R_h | C_C \wedge I_{co}), P(R_t | C_C \wedge I_{co})) = (1, 0) \quad (5)$$

for the outcomes.

C_G : Gwendolen, on the other hand, has no such particular skills, so if it were her who had tossed the coin we would assign the plausibility distribution

$$(P(R_i | C_G \wedge I_{co})) = (\frac{1}{2}, \frac{1}{2}), \quad (6)$$

with $i = h, t$ here and in the following.

C_J : On Jack we know nothing whatsoever. He could be skilled or unskilled in coin-tossing, a trickster or an absolutely earnest person. If we knew he had tossed the coin we could but assign the distribution

$$(P(R_i | C_J \wedge I_{co})) = (\frac{1}{2}, \frac{1}{2}). \quad (7)$$

C_A : Finally, we know that Algernon had been carrying a double-headed coin, which he would exchange with the original one if asked to toss it. So we assign the plausibilities

$$(P(R_i | C_A \wedge I_{co})) = (1, 0) \quad (8)$$

in case he had made the coin toss.

Remark 1. It is clear that not all the circumstances above express ‘causes’ [53] or ‘mechanisms’ [18, lects. 16, 17][19, lect. 5][20, ch. 18][78] which ‘determine’ the respective plausibility distributions. It could be appropriate to say this of the circumstance concerning Algernon; but the circumstance concerning Jack, e.g., can hardly be called a ‘cause’ or ‘mechanism’: it is only out of sheer ignorance that we assign, conditionally upon it, the distribution $(1/2, 1/2)$. Here and in the following, ‘circumstance’ will generally mean simply what its name denotes: ‘a possibly unessential or secondary condition, detail, part, state of affair, factor, accompaniment, or attribute, in respect of time, place, manner, agent, etc., that accompanies, surrounds, or possibly determines, modifies, or influences a fact or event’ (cf. [83]). \square

2.2 Grouping the circumstances in a special way

The crucial step now is the following. Suppose that these four circumstances interest us not for their intrinsic details, but only in connexion with the plausibility distributions they lead to for the coin toss in the context I_{co} . In this regard, the circumstance ‘Cecily tossed the coin’⁶ and the circumstance ‘Algernon tossed the coin’ are for us

⁶Our knowledge about Cecily must also be understood as implicit in this sentence; otherwise we should write ‘Cecily, who is a magician etc., tossed the coin’. This also holds for the sentences that follow.

equivalent, since both lead to the plausibility distribution $(1, 0)$, as shown by eqs. (5) and (8). Similarly, ‘Gwendolen tossed the coin’ and ‘Jack tossed the coin’ are also equivalent, both leading to $(1/2, 1/2)$; cf. eqs. (6) and (7). We should like to have a set of circumstances such that different circumstances led to different distributions. The first thing that comes to mind is to take the set $\{C_C \vee C_A, C_G \vee C_J\}$ of the disjunctions of equivalent circumstances, i.e. $C_C \vee C_A \equiv$ ‘Cecily or Algernon tossed the coin’ and $C_G \vee C_J \equiv$ ‘Gwendolen or Jack tossed the coin’. We must see, however, whether this ‘coarse-grained’ set really fulfils our wishes.

A simple theorem of plausibility theory comes to help. It says that, in a given context, the plausibility of a statement A conditional on a disjunction of mutually exclusive propositions $\{B_j\}$ is given by a convex sum of the plausibilities conditional on the single propositions, as follows [20, ch. 2]:

$$P[A | (\bigvee_j B_j) \wedge I] = \sum_j P[A | B_j \wedge I] \frac{P(B_j | I)}{\sum_l P(B_l | I)} \quad (\{B_j\} \text{ mutually exclusive}), \quad (9)$$

the weights being proportional to the plausibilities of the $\{B_j\}$. Note that the value of the plausibility conditional on the disjunction, $P[A | (\bigvee_j B_j) \wedge I]$, generally depends on the values of the plausibilities of the $\{B_j\}$, $\{P(B_j | I)\}$. Thus, the latter plausibilities must in general be specified if we want to find the first, and that varies as these vary. However, we see that this dependence disappears when the plausibilities conditional on each single B_j , $\{P[A | B_j \wedge I]\}$, have all the same value (the right-hand side becomes a convex sum of identical points). In this case also the plausibility conditional on the disjunction, $P[A | (\bigvee_j B_j) \wedge I]$, will have that same value, *irrespective of the plausibilities of the $\{B_j\}$* .⁷

if $P[A | B_j \wedge I] = q$ for all j , then $P[A | (\bigvee_j B_j) \wedge I] = q$,
regardless of the values of the $\{P(B_j | I)\}$. (10)

Clearly this is just the case when A is either of our outcomes $\{R_i\}$ and the $\{B_j\}$ are either pair of equivalent circumstances. In fact, the protasis of the last formula is just our previous definition of *equivalence* amongst circumstances. Hence, the plausibility distribution for the results conditional on the disjunction $C_C \vee C_A$ is the same as those conditional on the two disjuncts separately,

$$(P[R_i | (C_C \vee C_A) \wedge I_{co}]) = (P[R_i | C_C \wedge I_{co}]) = (P[R_i | C_A \wedge I_{co}]) = (1, 0), \quad (11)$$

and analogously for $C_G \vee C_J$:

$$(P[R_i | (C_G \vee C_J) \wedge I_{co}]) = (P[R_i | C_G \wedge I_{co}]) = (P[R_i | C_J \wedge I_{co}]) = (\frac{1}{2}, \frac{1}{2}). \quad (12)$$

This is true whatever the plausibilities of our four initial circumstances might be (in fact, we have not yet specified them!).

⁷Cases of vanishing plausibilities can be treated as appropriate limits. One can adopt the consistent convention that the product of an undefined plausibility (such as those with a contradictory context) times a defined and vanishing one also vanishes.

The coarse-grained set $\{C_C \vee C_A, C_G \vee C_J\}$ has thus, *by construction*, the special feature we looked for: different circumstances lead to different plausibility distributions for the outcomes. The circumstances can therefore be *uniquely indexed* by the respectively assigned plausibility distributions, and we denote them accordingly:

$$S_{(1,0)} := C_C \vee C_A, \quad S_{(\frac{1}{2},\frac{1}{2})} := C_G \vee C_J, \quad (13)$$

and call them *plausibility-indexed circumstances*. With this indexing system, and denoting $\mathbf{q} := (q_h, q_t)$, the conditional plausibilities of the outcomes can be written

$$P(R_i | S_{\mathbf{q}} \wedge I_{co}) = q_i. \quad (14)$$

The last expression is in many ways similar to that defining the generalised Bernoulli model (1). Indeed, one of our main points is the following: *plausibility-like parameters used as arguments of plausibilities can always be interpreted to stand for appropriate plausibility-indexed circumstances*.

In view of eq. (14), someone could interpret the symbol ‘ $S_{\mathbf{q}}$ ’ as ‘The plausibility distribution for the $\{R_i\}$ is \mathbf{q} ’ (similarly to the symbol ‘ A_p ’ introduced by Jaynes [18, lect. 18][19, lect. 5][20, ch. 18]). But such an interpretation is obviously wrong. Let us make this point clear. The symbol ‘ $S_{(1,0)}$ ’, e.g., stands for ‘Cecily or Algernon tossed the coin’, as eq. (13) shows; and this proposition does not concern plausibilities at all. It is true that this proposition is the *only* one leading us to assign the distribution $(1, 0)$; but it is so just because of a trick, viz. the fact that we have *grouped* and *indexed* the initial circumstances in a particular way. Borrowing some terminology from logic, we can say that the correspondence between the proposition ‘Cecily or Algernon tossed the coin’ and the distribution $(1, 0)$ is *only a trick within the metalanguage* of our theory [84, 85, 86, 87, 88].

Remark 2. The use of statements like ‘The plausibility of A is p ’ or ‘Data are drawn from a distribution f ’ is universal. Of course, they can be simply interpreted as ‘Look, the context and the circumstance are such that the plausibility of A (the data) is p (f)’, and this can be enough for our purposes: we may not need to know all the details of the context and the circumstance. But note that those statements are more precisely *metastatements*, statements *about* plausibility assignments. As in logic, the use of such kind of statements *as arguments of* plausibility formulae is preferably avoided. First, because such statements usually make poor contexts. Compare the statements ‘Either Jack, who is a skilled coin tosser with a predilection for ‘head’, or Algernon, who has a two-headed coin, tossed the coin’ with ‘The plausibility distribution for ‘head’ and ‘tail’ is $(1, 0)$ ’: the former gives some clues as to *the grounds on which* the distribution $(1, 0)$ is assigned, whereas the latter says only *that* that distribution is assigned.⁸ Second, because such statements used inside plausibility for-

⁸It reminds of Bachelierus’ oft quoted answer: “Mihi a docto Doctore/ Domandatur causam et rationem, quare/ Opium facit dormire?/ À quoi respondeo,/ Quia est in eo/ Virtus dormitiva./ Cujus est natura/ Sensus assoupire” [89, troisième intermède].

mulae may give rise to self-references, circularity, and thus known paradoxes (‘This proposition is false’) and other inconsistencies [88, 90].⁹ \square

2.3 Analysis by marginalisation

Let us now introduce the plausibilities of the original circumstances in the context I_{co} . For concreteness we can assume them to be equally plausible:

$$P(C_C | I_{\text{co}}) = P(C_G | I_{\text{co}}) = P(C_J | I_{\text{co}}) = P(C_A | I_{\text{co}}) = \frac{1}{4}. \quad (15)$$

From these values and the definitions (13) we have by the sum rule the plausibilities of the plausibility-indexed circumstances:

$$P[S_{(1,0)} | I_{\text{co}}] = \frac{1}{2}, \quad P[S_{(\frac{1}{2}, \frac{1}{2})} | I_{\text{co}}] = \frac{1}{2}. \quad (16)$$

These plausibilities can be used to write the distribution for the outcomes on context I_{co} by marginalisation over the circumstances. We can do this both with the initial set $\{C_j\}$ and with the set of plausibility-indexed set $\{S_q\}$. With the first we obtain

$$(P(R_i | I_{\text{co}})) = \sum_j (P(R_i | C_j \wedge I_{\text{co}})) P(C_j | I_{\text{co}}) = \left(\frac{3}{4}, \frac{1}{4}\right). \quad (17)$$

With the second set we must of course obtain, consistently, the same result; but the decomposition has a more suggestive (and possibly misleading!) form:

$$\begin{aligned} (P(R_i | I_{\text{co}})) &= \sum_q (P(R_i | S_q \wedge I_{\text{co}})) P(S_q | I_{\text{co}}), \\ &= \sum_q q P(S_q | I_{\text{co}}) = \left(\frac{3}{4}, \frac{1}{4}\right). \end{aligned} \quad (18)$$

The index q assumes the two values $\{(1,0), (1/2, 1/2)\}$, but we can let it range over the whole simplex Δ defined in (2), introducing a density function $q \mapsto p_S(q | I_{\text{co}})$ in the usual way (explained later in § 4). In this case it is given by

$$p_S(q | I) := \frac{1}{2} \left[\delta(q_h - 1) + \delta\left(q_h - \frac{1}{2}\right) \right] \delta(q_h + q_t - 1), \quad (19)$$

⁹We find an example in an article by Friedman and Shimony [91]. They introduce a proposition which says that the expectation of a certain quantity has a given value (their eq. (4)). But such a proposition is a metastatement, because *expectation* is defined in terms of plausibility assignments (in contrast to *average*, which is defined in terms of measured frequencies [92]). The authors, however, do not notice this and proceed to use that proposition inside plausibilities, obtaining peculiar conclusions. Gage and Hestenes [93] apparently show that these conclusions are not inconsistent, although they do not notice the mix-up of language and metalanguage either. Cyranski [94] has a partially clearer view of the matter. Cf. remark 6. A metastatement inside a plausibility is used, although tentatively, also by Jaynes (his ‘ A_p ’) [18, lect. 18][19, lect. 5][20, ch. 18]; but our analysis shows that his ideas can be realised without this artifice.

a weighted sum of Dirac deltas¹⁰ with support on $\mathbf{q} = (1, 0)$ and $(1/2, 1/2)$. The marginalisation (18) thus takes the form

$$\left(P(R_i | I_{\text{co}})\right) = \int_{\Delta} \mathbf{q} p_S(\mathbf{q} | I_{\text{co}}) d\mathbf{q}, \quad (20)$$

which is similar to the formula (3) for the generalised Bernoulli model.

2.4 Updating the plausibility of the circumstances

If the outcome of the toss is, say, ‘head’, what do the plausibilities of the circumstances become? In other words, what are the circumstances’ plausibilities in the context $R_h \wedge I_{\text{co}}$? The answer is obviously given by Bayes’ theorem:

$$P(S_q | R_h \wedge I_{\text{co}}) = \frac{P(R_h | S_q \wedge I_{\text{co}}) P(S_q | I_{\text{co}})}{\sum_{q'} P(R_h | S_{q'} \wedge I_{\text{co}}) P(S_{q'} | I_{\text{co}})}, \quad (21)$$

or, in terms of the density p_S ,

$$p_S(\mathbf{q} | R_h \wedge I_{\text{co}}) = \frac{q_h p_S(\mathbf{q} | \wedge I_{\text{co}})}{\int_{\Delta} q'_h p_S(\mathbf{q}' | \wedge I_{\text{co}}) d\mathbf{q}'} = \left[\frac{2}{3} \delta(q_h - 1) + \frac{1}{3} \delta\left(q_h - \frac{1}{2}\right) \right] \delta(q_h + q_t - 1). \quad (22)$$

The plausibility of $S_{(1,0)}$, i.e., that Cecily or Algernon tossed the coin, has thus increased a little.

Remark 3. Note that knowledge of the outcome can help to increase the plausibility of one of the plausibility-indexed circumstances $\{S_q\}$ at the expense of the others’, but can never do so within a set of *equivalent* circumstances like $\{C_C, C_A\}$ or $\{C_G, C_J\}$. \square

The last formula is a very simple instance of the answer to an inverse problem. Our point is, again, that the marginalisation over a plausibility-like parameter and the updating of its distribution can be interpreted as the same operations for a set of plausibility-indexed circumstances. From this standpoint, and as should be clear from a previous discussion and remarks, the plausibility $P(S_q | I_{\text{co}})$ (and its density $p_S(\mathbf{q} | I_{\text{co}})$) is not the plausibility of a plausibility, but simply the plausibility of a circumstance, the latter being indexed in a particular way.

3 Second Example: multiple measurements, particular convex structures of circumstances, updating

3.1 Context

The following example differs from the first in the number of measurements and circumstances considered. Consequences: the space of parameters has particular

¹⁰In Egorov’s sense [95, 96, 97]; see also [98, 99, 100, 101] and cf. [102, 103, 104].

convex structures, and the plausibilities of the outcomes of one measurement can be ‘updated’ upon knowledge of the outcome of the other.

We have a box with two buttons, marked ‘Letter’ and ‘Number’, and a display. Push the ‘Letter’ button, and either ‘a’ or ‘b’ appears on the display; push ‘Number’, and ‘1’ or ‘2’ appears. We can push each button only once, and only one at a time. Call, improperly, *measurement* the act of pushing a button and reading the display; call ‘outcome’ what is then read on the display. Denote the ‘Letter’ measurement by M^L and its outcomes by $\{R_a^L, R_b^L\}$; the ‘Number’ measurement by M^N and its outcomes by $\{R_1^N, R_2^N\}$.

Given only the above knowledge, we should assign a plausibility distribution $(1/2, 1/2)$ to the outcomes of each measurement. But we know in fact something more about the construction of the box: inside, besides some sort of machinery, there is a chest containing an even number, $2N$, of balls. Each ball is marked either ‘a1’, ‘a2’, ‘b1’, or ‘b2’. When a button is pushed, the machinery draws one of the balls from the chest and sends, depending on the button, either the letter *or* the number printed on the drawn ball to the display; and then puts the ball back into the chest.¹¹ We have also a very important piece of knowledge as to how the $2N$ balls were originally chosen and put into the chest: this initially contained $4N$ balls, marked ‘a1’, ‘a2’, ‘b1’, and ‘b2’ in equal proportions (i.e., N balls marked ‘a1’, N ‘a2’, etc.). From these, $2N$ balls were taken away, so only $2N$ remained in the chest. This is all we know; denote it (together with everyday knowledge concerning balls, buttons, boxes, etc.) by I_N .

From I_N , some points are immediately clear. First, not all the $2N$ balls in the chest can be marked ‘a1’, nor all ‘a2’, etc., since the chest initially contained only N of each type. Second, if all the $2N$ balls have the ‘a’ mark, then N of them must necessarily be of the ‘a1’ kind and the other N must be of the ‘a2’ kind. Similarly for the marks ‘b’ and, exchanging the rôle of letters and numbers, ‘1’ and ‘2’.

3.2 Introducing a set of circumstances

Let us analyse the context I_N into a set of mutually exclusive and exhaustive possible circumstances. Different choices are possible. One is to consider the possible sets of balls left in (or equivalently, taken away from) the chest. The number of circumstances thus defined is given by the number of ways of choosing $2N$ objects from a collection of $4N$ distinct ones without regard to order — the binomial coefficient $\binom{4N}{2N}$. Note that it matters *which* of the ‘a1’-marked balls are chosen, and likewise for the others. Our knowledge is symmetric in respect of these circumstances, hence they are assigned equal plausibilities.¹²

¹¹This renders the temporal order of the measurements (if both are performed) irrelevant. That is why we are not making temporal considerations. (Note also that we do not need to suppose that the urn is shaken after the replacement of the ball: this would add nothing to our state of knowledge, since we do not know how the machine makes the replacement anyhow.)

¹²That is, they are assigned equal plausibilities not because ‘the balls are initially chosen at random’ or something of the kind, but because we just do not know how they have been chosen. In fact, they

Another choice is to consider as a circumstance the numbers of balls marked ‘a1’, ‘a2’, etc. left in the chest instead. Note the difference with the previous choice: this is a sort of ‘coarse graining’ thereof. For this reason the newly defined circumstances are not equally plausible. We settle for this second choice and denote a generic circumstance by $C_{\alpha a1, \beta a2, \gamma b1, \delta b2}$, meaning ‘ α ‘a1’-marked balls, ..., and δ ‘b2’-marked balls are left in the chest’. The coefficients α, β , etc. must obviously sum up to $2N$ and each can range from 0 to N .

3.3 Plausibility-indexing the circumstances; their particular set

As in the previous example, suppose that we are not interested in the details of the circumstances above, but only in the plausibilities they lead us to assign to the outcomes of the two measurements M^L and M^N . We can group the circumstances into plausibility-indexed equivalence classes, as before. In the present case the equivalence must take into account *two* plausibility distributions, one for each measurement.

Here is an example for $N = 2$. The two different circumstances $C_{(2a1, 0a2, 1b1, 1b2)}$ and $C_{(1a1, 1a2, 2b1, 0b2)}$ lead both to the same plausibility distribution $(1/2, 1/2)$ for the ‘Letter’ measurement, and to the same distribution $(3/4, 1/4)$ for the ‘Number’ measurement (as is clear by simply counting their ‘a’s, ‘b’s, ‘1’s, and ‘2’s). Moreover, *only* these two circumstances lead to the plausibility distributions above, as the reader can prove. By theorem (10), also their disjunction $C_{(2a1, 0a2, 1b1, 1b2)} \vee C_{(1a1, 1a2, 2b1, 0b2)}$ leads to the same distributions and can thus be denoted by

$$S_{((1/2, 1/2), (3/4, 1/4))} := C_{(2a1, 0a2, 1b1, 1b2)} \vee C_{(1a1, 1a2, 2b1, 0b2)}. \quad (23)$$

This is one of the plausibility-indexed circumstances. Its plausibility is the sum of its disjuncts’ plausibilities, $P[S_{((1/2, 1/2), (3/4, 1/4))} | I_N] = P(C_{(2a1, 0a2, 1b1, 1b2)} | I_N) + P(C_{(1a1, 1a2, 2b1, 0b2)} | I_N)$.

In general, for any N , we have plausibility-indexed circumstances denoted by $S_{(q^L, q^N)}$, the parameters q^L and q^N corresponding to the plausibility distributions for the ‘Letter’ and the ‘Number’ measurements. The indexing is such that

$$P(R_i^k | M^k \wedge S_{(q^L, q^N)} \wedge I_N) = q_i^k \quad \text{for } k = L, N \text{ and all appropriate } i. \quad (24)$$

We leave to the reader the pleasure of proving that there is a total of $N^2 + (N + 1)^2$ plausibility-indexed circumstances, i.e., of distinct values for the parameters (q^L, q^N) . They can be represented by points on the plane $q_a^L q_1^N$ as illustrated in fig. 1 for the cases $N = 1$, $N = 4$, and $N = 16$ respectively. It is not difficult to see (especially looking at the figure for $N = 16$) that as $N \rightarrow \infty$ their set Γ_N becomes dense in the convex set Γ_∞ defined by

$$\Gamma_\infty := \{(q^L, q^N) \mid \|q^L\|_\infty + \|q^N\|_\infty \leq 1\}, \quad (25)$$

can have been chosen according to a particular scheme; the point is that we do not know such scheme.

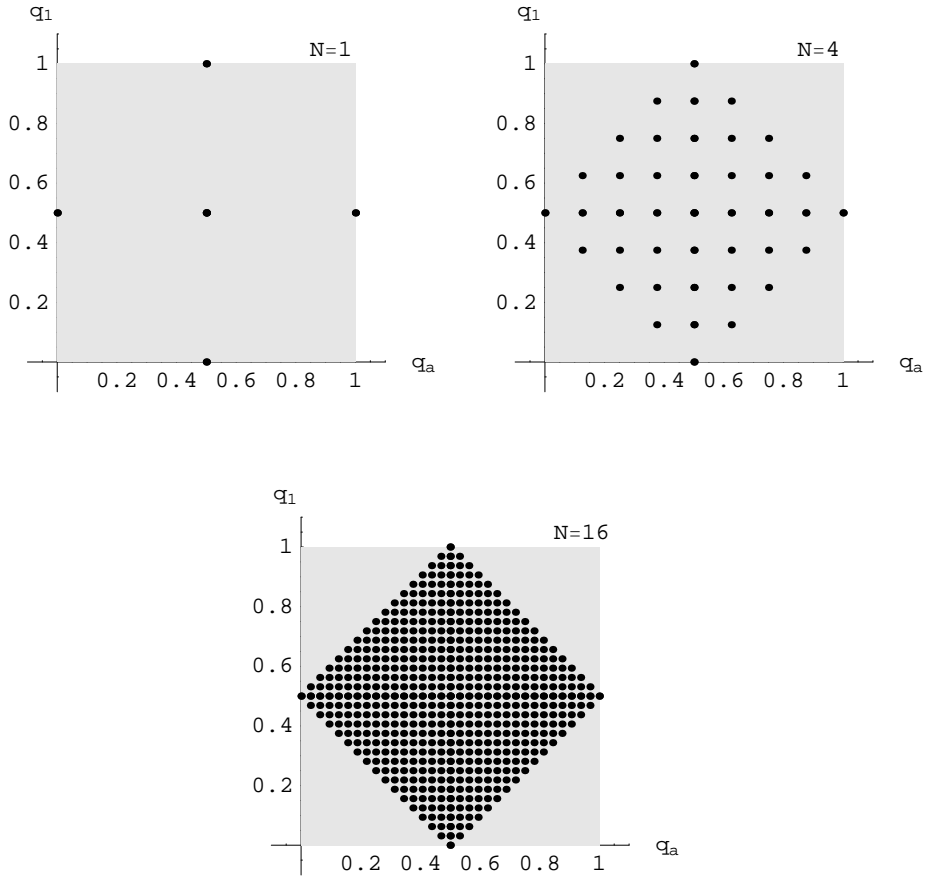


Figure 1: Possible values of the plausibilities q_a^L and q_1^N for $N = 1, 4$, and 16 . They are in bijective correspondence with the plausibility-indexed circumstances. The grey region is the set of all possible pairs of plausibility distributions for two generic measurements.

where $\|q\|_\infty$ is the supremum norm $\|q\|_\infty := \max_i \{q_i\}$. Thus in the limit $N \rightarrow \infty$ we may effectively work with a continuum of plausibility-indexed circumstances in bijection with the points of this set. Denote this ‘limit context’ by I_∞ .

We observe two interesting facts. The first is that Γ_∞ is a *proper* subset of the set of all possible pairs of plausibility distributions for two generic measurements (the grey square region in the figure); the latter is the Cartesian product of two one-dimensional simplices, $\Delta_1 \times \Delta_1$. We could have let the parameters (q^L, q^N) range over the latter set; in this case, however, the contexts I_N and I_∞ would have led us to a vanishing plausibility density for those parameter values not belonging to Γ_∞ .

The second interesting fact is that neither Γ_∞ nor the larger set $\Delta_1 \times \Delta_1$ are simplices. It is so because we are considering *two* measurements (had we considered a single measurement with four outcomes, we would have dealt with a three-dimensional simplex instead).¹³ See also remark 7.

3.4 Analysis by marginalisation

We can write the plausibility distributions for the measurements as marginalisations over the plausibility-indexed circumstances $\{S_{(q^L, q^N)}\}$, using the latter’s plausibilities $\{P[S_{(q^L, q^N)} | I_N]\}$. Denote for brevity $(q^L, q^N) =: \bar{q}$, hence $\{S_{(q^L, q^N)}\} \equiv \{S_{\bar{q}}\}$. Then

$$\begin{aligned} (P(R_i^k | M^k \wedge I_N)) &= \sum_{\bar{q} \in \Gamma_N} (P(R_i^k | M^k \wedge S_{\bar{q}} \wedge I_N)) P(S_{\bar{q}} | I_N), \\ &= \sum_{\bar{q} \in \Gamma_N} q^k P(S_{\bar{q}} | I_N), \end{aligned} \quad k = L, N \quad (26)$$

Also this formula, like (3) and (18), looks like a weighted sum of plausibilities, and $P(S_{\bar{q}} | I_N)$ looks like the plausibility of two plausibility distributions. But this is not the case, just as it was not in the example of the coin: the propositions $\{S_{\bar{q}}\}$ speak not about plausibilities but about possible preparations of the box and its contents; yet they are suitably indexed according to the plausibilities they lead us to assign to the measurements’ outcomes.

The sums above can also be replaced by integrals over the set Γ_∞ ,

$$(P(R_i^k | M^k \wedge I_N)) = \int_{\Gamma_\infty} q^k p_S(\bar{q} | I_N) d\bar{q}, \quad (27)$$

where the density $\bar{q} \mapsto p_S(\bar{q} | I_\infty)$ is introduced just like in the example of the coin.

3.5 ‘Updating’ the plausibilities of the circumstances and the outcomes

Suppose that the ‘Letter’ button has been pushed and the outcome ‘a’ has appeared on the display. This knowledge places us in a new context, expressed by the proposition $M^L \wedge R_a^L \wedge I$. We ask: (1) What plausibilities

$$P(S_{\bar{q}} | R_a^L \wedge M^L \wedge I_N) \quad (28)$$

do we assign to the plausibility-indexed circumstances $\{S_{(q^L, q^N)}\}$ in the new context? Furthermore, we still have the possibility of pushing the ‘Number’ button once. So we also ask: (2) What plausibilities

$$P(R_i^N | M^N \wedge R_a^L \wedge M^L \wedge I_N) \quad (29)$$

¹³You might ask: “Couldn’t we consider a single measurement with the four outcomes ‘a1’, ‘a2’, ‘b1’, ‘b2’ instead?”. The answer is: yes, we could have introduced a single *fictional* measurement with M^L and M^N arising as marginals. But what for? After all, the rules of this game do not make allowance for such a measurement.

do we now assign to the outcomes $\{R_1^N, R_2^N\}$ in case we push the ‘Number’ button?

Let us answer the first question. We use the assumption (valid in the context I_N) that knowledge of the performance of any of the two measurements (but not of their outcomes!) is irrelevant for assigning plausibilities to the circumstances:

$$P(S_{\bar{q}} | M^k \wedge I_N) = P(S_{\bar{q}} | M^L \wedge M^N \wedge I_N) = P(S_{\bar{q}} | I_N) \quad \text{for all } \bar{q}, k. \quad (30)$$

With this assumption and eq. (24), Bayes’ theorem yields a simplified form for the sought plausibilities (28):

$$P(S_{\bar{q}} | R_a^L \wedge M^L \wedge I_N) = \frac{q_a^L P(S_{\bar{q}} | I_N)}{\sum_{\bar{q}'} q_a'^L P(S_{\bar{q}'} | I_N)}. \quad (31)$$

This is also the answer to an inverse problem. Note, again, that it expresses the updated plausibility distribution, not ‘of the parameter \bar{q} ’, but of propositions like ‘There are α ‘a1’-marked balls, ..., and δ ‘b2’-marked balls left in the chest; or ...; or α' ‘a1’-marked balls, ..., and δ ‘b2’-marked balls left in the chest’.

To answer the second question we use, beside assumption (30), the following fact, which holds in our context I_N : If we want to determine the plausibility distribution for one of the measurements, and we know which particular circumstance holds, then for us it is irrelevant to know whether the other measurement has been performed, or which outcome it has yielded. For example, if we are interested in the plausibilities of the outcomes of the ‘Number’ measurement, and we know that a particular circumstance $S_{\bar{q}}$ holds (e.g., that in the chest there are two ‘a1’-marked balls, one ‘a2’-marked ball, etc.; or one ‘a1’-marked ball, one ‘a2’-marked ball, etc.), then knowledge of the outcome of the mere performance of ‘Letter’ measurement is irrelevant. In formulae,

$$\begin{aligned} P(R_i^k | (R_j^l \wedge M^l) \wedge M^k \wedge S_{\bar{q}} \wedge I_N) &= P(R_i^k | M^l \wedge M^k \wedge S_{\bar{q}} \wedge I_N) = \\ &= P(R_i^k | M^k \wedge S_{\bar{q}} \wedge I_N) \quad \text{for all } k, l \neq k, \text{ and appropriate } i, j. \end{aligned} \quad (32)$$

Analysing eq. (29) in terms of circumstances and using eqs. (31) and (32) we find

$$P(R_i^N | M^N \wedge R_a^L \wedge M^L \wedge I_N) = \frac{\sum_{\bar{q}} q_i^N q_a^L P(S_{\bar{q}} | I_N)}{\sum_{\bar{q}'} q_a'^L P(S_{\bar{q}'} | I_N)}. \quad (33)$$

In regard to the assumptions summarised in eqs. (30) and (32), cf. remark 9.

4 Generalisation and summary of principal formulae

The two examples should suffice to give an idea of the interpretation of \bar{q} -like parameters and of their plausibilities, and of the principal consequences of this interpretation. The reader could try to make similar analyses for the toy models by Kirkpatrick [105, 106, 107], Spekkens [108], or us [109, 110]. We shall now present the idea in general and abstract terms. Some additional remarks will also be given.

4.1 Experiments, outcomes, circumstances

In the general case we have a context I and a set of m *measurements*, represented by propositions M^k , $k = 1, \dots, m$. Each measurement M^k has mutually exclusive and exhaustive *outcomes* represented by a set of propositions $\{R_i^k\}$. The number of outcomes can vary from measurement to measurement, so that i ranges over appropriate sets for different k . The index k is omitted when no confusion arises.

Remark 4. The use of the terms ‘measurement’ and ‘outcome’ is only dictated by concreteness. The formalism and the discussion presented apply in fact to more general concepts. What we call ‘measurement’ could be only a casual observation, or simply a ‘state of affairs’ which can present itself in mutually exclusive and exhaustive ‘forms’ (the ‘outcomes’). The term ‘measurement’ shall hence be divested here of those connotations implying active planning and control, which are not relevant to our study. Moreover, a ‘measurement’ needs not be associated with a point or short interval in time or space. It can e.g. be a collection of observations; in this case its ‘outcomes’ are all possible combinations of results from these observations. Finally, note that the m measurements are generally different, i.e., they are not necessarily ‘repetitions’ of the ‘same’ measurement — a case that will be discussed in the second paper instead. \square

A set of *circumstances* $\{C_j\}$ is introduced; these represent a sort of more detailed, possible descriptions of the context I , and are mutually exclusive and exhaustive, i.e. we know that one and only one of them holds:

$$P(C_{j'} \wedge C_{j''} | I) = 0 \quad \text{for all } j' \text{ and } j'' \neq j', \quad (34)$$

$$P(\bigvee_j C_j | I) = 1. \quad (35)$$

The plausibilities of the measurements’ outcomes conditional on the circumstances,

$$P(R_i | M^k \wedge C_j \wedge I) \quad \text{for all } j, k, \text{ and appropriate } i, \quad (36)$$

are assumed to be given.

Remark 5. The notion of ‘circumstance’, represented by propositions C_j and later also S_q , has been further explained in remark 1. An example of circumstance from § 2 is ‘Gwendolen tossed the coin’; other examples are ‘The temperature during the experiment was 25 °C’ and the more elaborated ‘We studied the density of monodisperse spherical particles in a tall cylindrical tube as a series of external excitations, consisting of discrete, vertical shakes or ‘taps,’ were applied to the container’ [111]. As in the case of ‘measurement’, a circumstance needs not be related to a single point or short interval in space or time. For example, in assigning the plausibility that it will rain or has rained in a given place at a given time, a circumstance might consist in a specific history of worldwide meteorological conditions under the preceding two years. For reasons discussed in remark 2, we require that a circumstance be described or specified in concrete terms, and metastatements like ‘The samples are drawn from a distribution f ’ or ‘The plausibility of head is 1/3’ are excluded. Finally, the choice

of an appropriate set of circumstances, i.e., of the appropriate way and depth to analyse a particular problem (the context), can only be decided on an individual basis, of course. \square

4.2 Plausibility-indexing the circumstances

The circumstances are then grouped into equivalence classes. Two circumstances are equivalent if they lead to the same plausibility distributions for each measurement M^k :

$$C_{j'} \sim C_{j''} \iff \begin{cases} P(R_i | M^k \wedge C_{j'} \wedge I) = P(R_i | M^k \wedge C_{j''} \wedge I) \\ \text{for all } k \text{ and appropriate } i. \end{cases} \quad (37)$$

By construction the equivalence classes are in injective correspondence with the possible numerical values of the plausibility distributions for the measurements, (q^1, q^2, \dots) . Denote a generic such value by $\bar{q} := (q^k)$, its equivalence class by \bar{q}^\sim , and membership by $C_j \in \bar{q}^\sim$ or simply $j \in \bar{q}^\sim$. We take all disjunctions of equivalent circumstances

$$S_{\bar{q}} := \bigvee_{j \in \bar{q}^\sim} C_j, \quad (38)$$

and call these (in lack of a better name) *plausibility-indexed circumstances*, shortened to ‘circumstances’ whenever no confusion is possible. Conditional on such a circumstance $S_{\bar{q}}$, the plausibilities of the outcomes have numerical values identical to its indices:

$$P(R_i^k | M^k \wedge S_{\bar{q}} \wedge I) = q_i^k, \quad (39)$$

a formula that reminds of a generalised Bernoulli model (cf. eq. (1)).

Our main belief, already stated in the coin example, is that *plausibility-like parameters used as arguments of plausibilities can always be interpreted to stand for some appropriate plausibility-indexed circumstances*.¹⁴

The passage to plausibility-grouped circumstances can have two main motivations. (1) We can be interested in the plausibilities the circumstances lead to, rather than in the latter’s intrinsic details. (2) We may want a set of circumstances with the property that knowledge of outcomes can increase the plausibility of only one circumstance. This is true for the set $\{S_{\bar{q}}\}$, but not for the set $\{C_j\}$ in general. In fact, *knowledge of new outcomes can never lead to a alteration of the ratios of the plausibilities of two or more equivalent circumstances*. Cf. remark 3 and see [1, 2] and [46, § 5.3].

¹⁴What constitutes a circumstance is largely a matter of situation, purpose, and personal good taste as well. The formalism presented cannot think up the circumstances for us. In § 2 we spoke e.g. about different persons’ skills in coin-tossing; but other people could speak about different values of the coin’s ‘propensity’ to come up heads. Perhaps the reason why ‘de Finettians’ have always felt uneasy about plausibility-like parameters and their priors was that these mathematical objects leave room to ideas and concepts that are unnecessary or not in good taste (cf. Jaynes [20], ch. 3, ‘Logic versus propensity’). To keep off these ideas they partially denied priors their meaning as plausibilities (this has led, fortunately, to some very beautiful ideas and theorems [64]). We hope to have shown here and in the next paper that there is no need to adopt such extreme measures.

Remark 6. Suppose that to each outcome R_i of some measurement M_k is associated a value r_i of some physical quantity, so that it makes sense to speak of the *expected value*¹⁵ of this quantity in a generic context $M_k \wedge J$:

$$\langle r | M_k \wedge J \rangle := \sum_i r_i P(R_i | M_k \wedge J). \quad (40a)$$

In our case, the formation of equivalence classes of circumstances can then be made with respect to expected values instead of plausibilities, i.e.,

$$C_{j'} \sim C_{j''} \iff \langle r^k | M^k \wedge C_{j'} \wedge I \rangle = \langle r^k | M^k \wedge C_{j''} \wedge I \rangle \text{ for all } k. \quad (40b)$$

In this way we obtain a set of *expectation-indexed* circumstances $\{S_{\langle r \rangle}\}$. Note that two different circumstances in such a set (leading hence to different expectations) may lead to the *same* probability distributions for the outcomes; therefore this set is not to be confused with, and has not the same applications of, our $\{S_{\bar{q}}\}$. \square

Particularly interesting is the space Γ of the parameters \bar{q} . Since these correspond to numerical values of plausibility distributions for the measurements, Γ is in general a (possibly proper) subset of a Cartesian product of simplices $\times_k \Delta^{(k)}$, the simplex $\Delta^{(k)}$ corresponding to the plausibility distribution for the k th measurement.

Remark 7. The features of the subset Γ will depend on the nature of the circumstances $\{C_j\}$ (and thus of the $\{S_{\bar{q}}\}$). In some cases it is simply postulated that some kinds of circumstances do not present themselves, and this will delimit the subset accordingly. We saw an instance of this in the box example of § 3, in which the set Γ was, for each N , a special proper subset Γ_N of the Cartesian product of two two-dimensional simplices (the grey square region in the figures). There are examples of physical theories where we postulate (by induction from numerous observations) that the set of ‘circumstances in which a system is prepared’ — often called *states* — is somehow restricted. This also restricts the space of the mathematical objects representing these states to particular, non-simplicial (convex) sets. The most notable example is quantum theory, in which the set of statistical operators — the mathematical objects representing the states — has very strange shapes [112, 113, 114, 115].¹⁶ The set of Gibbs distributions in classical statistical mechanics provides another example. \square

Remark 8. The plausibility-indexed circumstances need not be parametrised by the values of the plausibility distributions ($q^k \equiv \bar{q}$). Other parametrisations can be used as long as they are in bijective correspondence with the \bar{q} one, and some may be more useful (cf. [116]). Usually, what is relevant is the convex structure of the set of parameters Γ , a point on which we shall return in the third paper. \square

¹⁵Which should not be confused with the *average* [92], defined in terms of observed frequencies. Cf. footnote 9.

¹⁶That is, if we represent this set so as to preserve its convex properties, which are the relevant ones (see the third note of this series).

4.3 Priors and analysis by marginalisation

If the initial circumstances $\{C_j\}$ have the plausibility distribution $(P(C_j|I))$, by the sum rule the plausibility-indexed circumstances have distributions

$$P(S_{\bar{q}}|I) = \sum_{j \in \bar{q}} P(C_j|I) \quad (41)$$

(see also remark 11).

In terms of the plausibility-indexed circumstances, the plausibility distribution for each measurement outcome R_i^k can be expressed in marginal form as

$$\begin{aligned} P(R_i^k | M^k \wedge I) &= \sum_{\bar{q} \in \Gamma} P(R_i^k | M^k \wedge S_{\bar{q}} \wedge I) P(S_{\bar{q}} | I), \\ &\equiv \sum_{\bar{q} \in \Gamma} q_i^k P(S_{\bar{q}} | I), \\ &\equiv \int_{\bar{\Gamma}} q_i^k p_S(\bar{q} | I) d\bar{q}, \end{aligned} \quad (42)$$

(cf. eq. (3)) where $\bar{q} \mapsto p_S(\bar{q} | I)$ is an appropriate generalised function [98, 117, 118, 119] (see also [120, 121]). The sudden appearance of an integral can be justified (as customary) as follows: \bar{q} becomes a continuous parameter whose range is some set $\bar{\Gamma}$ such that $\text{conv } \Gamma \subseteq \bar{\Gamma} \subseteq \times_k \Delta^{(k)}$ (where $\text{conv } \Gamma$ is the convex hull of Γ), and we introduce a density function $\bar{q} \mapsto p_S(\bar{q} | I)$ such that, for each $\omega \subseteq \bar{\Gamma}$ (from a suitable σ -field of subsets), $\int_{\omega} p_S(\bar{q} | I) d\bar{q} = \sum_{\bar{q}' \in \omega \cap \Gamma} P(S_{\bar{q}'} | I)$.¹⁷

Note that to obtain the marginal form above it is assumed that knowledge of the measurement performed (but not of its outcome!) is irrelevant for assigning the plausibilities to the circumstances (cf. eq. (30)):

$$P(S_{\bar{q}} | M^{k_1} \wedge \dots \wedge M^{k_n} \wedge I) = P(S_{\bar{q}} | I) \quad \text{for all } \bar{q}, n = 1, \dots, m, \text{ and } \{k_t\}. \quad (43)$$

4.4 Updating the plausibilities of circumstances and outcomes

Upon knowledge of the outcomes $\{R_{i_1}^{k_1}, \dots, R_{i_n}^{k_n}\}$ of any subset $\{M^{k_1}, \dots, M^{k_n}\}$, $n \leq m$, of measurements, the $\{k_t\}$ being all mutually different, the plausibilities of the circumstances are updated, with the assumption (43), according to

$$P[S_{\bar{q}} | (R_{i_1}^{k_1} \wedge M^{k_1}) \wedge \dots \wedge (R_{i_n}^{k_n} \wedge M^{k_n}) \wedge I] = \frac{q_{i_1}^{k_1} \dots q_{i_n}^{k_n} P(S_{\bar{q}} | I)}{\sum_{\bar{q}' \in \Gamma} q_{i_1}^{k_1} \dots q_{i_n}^{k_n} P(S_{\bar{q}'} | I)}, \quad (44)$$

¹⁷No one forbids us to introduce additional *impossible* fictitious circumstances $\{C_{j'}\}$ (which may involve, e.g., ‘centaurs, nectar, ambrosia, fairies’ [122]) constructed so as to ‘complete’ the set of plausibility-indexed circumstances, i.e., in such a way that for each $\bar{q} \equiv (q_i^k) \in \bar{\Gamma}$ (note the bar!) there is always an $S_{\bar{q}}$ — possibly defined in terms of the fictitious $\{C_{j'}\}$ — for which $P(R_i | M^k \wedge S_{\bar{q}} \wedge I) = q_i^k$. This operation — which is, mark, not necessary — has no importance nor mathematical consequences because the fictitious circumstances are impossible, i.e., their plausibilities in the context I are naught, and thus terms containing them give no contribution in formulae like (42) or (47).

or, in terms of the density p_S ,

$$p_S[\bar{q} | (R_{i_1}^{k_1} \wedge M^{k_1}) \wedge \dots \wedge (R_{i_n}^{k_n} \wedge M^{k_n}) \wedge I] = \frac{q_{i_1}^{k_1} \dots q_{i_n}^{k_n} p_S(\bar{q} | I)}{\int_{\bar{q}} q_{i_1}^{k_1} \dots q_{i_n}^{k_n} p_S(\bar{q}' | I) d\bar{q}'}. \quad (45)$$

These formulae are valid for $n \geq 2$ only if we assume that, when a circumstance is known and we want to assign a plausibility distribution for a measurement, knowledge of performance of other measurements or of their outcomes is irrelevant (this is what Caves calls, in a slightly different context (see the second paper in this series), ‘learning through the parameter’ [78]):

$$P(R_i^k | E \wedge M^k \wedge S_{\bar{q}} \wedge I) = P(R_i^k | M^k \wedge S_{\bar{q}} \wedge I) \quad (46)$$

for all \bar{q} , where E is any conjunction of any number of mutually different $\{M^{k_i}\}$ and any number of $\{R_{i_i}^{k_i}\}$ (each $k_i \neq k$).

Under the assumptions (43) and (46), we also obtain, by marginalisation over the $\{S_{\bar{q}}\}$, the plausibility of an outcome R_i^k given knowledge of outcomes of other measurements $\{M^{k_i}\}$ different from M^k :

$$P(R_i^k | (R_{i_1}^{k_1} \wedge M^{k_1}) \wedge \dots \wedge (R_{i_n}^{k_n} \wedge M^{k_n}) \wedge M^k \wedge I) = \frac{\int_{\bar{q}} q_i^k q_{i_1}^{k_1} \dots q_{i_n}^{k_n} p_S(\bar{q} | I) d\bar{q}}{\int_{\bar{q}} q_{i_1}^{k_1} \dots q_{i_n}^{k_n} p_S(\bar{q}' | I) d\bar{q}'}. \quad (47)$$

Remark 9. We should always be careful in assuming and using the conditions summarised in eqs. (43) and (46), because they in many cases do not hold. An example would be provided by the example of the coin toss if we considered other tosses made by the same, unknown, person. In the circumstance in which Jack tosses the coin, eq. (46) would not hold because from the results of other tosses we would learn more about Jack’s skills in coin-tossing. In fact, even eq. (7) could cease to be valid for other tosses, and our set of circumstances would no longer be appropriate. We discuss similar matters in more detail in the second part of this study. In general, also the relations amongst the times or places at which measurements are performed can be relevant and thus require a careful analysis. Cf. the examples in refs. [105, 106, 107, 108, 109]. \square

4.5 Further remarks

Remark 10. A very important point is that the analysis of the context in terms of circumstances is far from unique (cf. footnote 14). Different sets $\{C'_{j'}\}$, $\{C''_{j''}\}$, $\{C'''_{j'''}\}$, etc. of circumstances can be introduced to analyse the context, and from them corresponding sets of plausibility-indexed circumstances $\{S'_{\bar{q}} \mid \bar{q} \in \Gamma'\}$, $\{S''_{\bar{q}} \mid \bar{q} \in \Gamma''\}$, $\{S'''_{\bar{q}} \mid \bar{q} \in \Gamma'''\}$, etc. can be constructed in the standard way. The circumstances of each set have to be mutually exclusive and exhaustive for the present formalism to hold, but they need not be exclusive with those of the other sets. For example, in

the case of the coin toss (§ 2) we could analyse the context I_{co} into another set of circumstances, say $\{C'_r \mid r \in]-1, 1[\}$ with

$$C'_r := \text{‘The mass-centre of the coin lies on the coin’s (oriented) axis a fraction } r/2 \text{ of the total width away from the coin centre’} \quad (48)$$

The analysis and the construction of the plausibility-indexed circumstances would proceed exactly in the same way, apart from possibly different values of their plausibilities.¹⁸

Different sets $\{C'_j\}, \{C''_{j'}\}, \dots$ can also be combined into a single set with circumstances $\{C_{j'j''\dots} := C'_{j'} \wedge C''_{j''} \wedge \dots\}$. These will be mutually exclusive and exhaustive by construction. Again, the corresponding plausibility-indexed set $\{S_{\bar{q}}\}$ will ensue in the usual way. \square

Remark 11. In view of the preceding remark it is clear that we can find *a* meaning for a plausibility-parameter like \bar{q} in terms of a set of circumstances, but not *the* meaning, because that set is not unique. This also implies that different choices of priors for \bar{q} need not be contradictory, because they can arise as the plausibilities for different sets of circumstances. There are, however, some compatibility conditions that the plausibility distributions for two or more sets of plausibility-indexed circumstances must satisfy (here stated in terms of densities):

$$\begin{aligned} \int_{\bar{F}'} q_{i_1}^{k_1} p_{S'}(\bar{q} \mid I) d\bar{q} &= \int_{\bar{F}''} q_{i_1}^{k_1} p_{S''}(\bar{q} \mid I) d\bar{q}, \\ \int_{\bar{F}'} q_{i_1}^{k_1} q_{i_2}^{k_2} p_{S'}(\bar{q} \mid I) d\bar{q} &= \int_{\bar{F}''} q_{i_1}^{k_1} q_{i_2}^{k_2} p_{S''}(\bar{q} \mid I) d\bar{q}, \\ &\dots \\ \int_{\bar{F}'} q_{i_1}^1 q_{i_2}^2 \dots q_{i_m}^m p_{S'}(\bar{q} \mid I) d\bar{q} &= \int_{\bar{F}''} q_{i_1}^1 q_{i_2}^2 \dots q_{i_m}^m p_{S''}(\bar{q} \mid I) d\bar{q}, \end{aligned}$$

for all mutually different k_t and appropriate i_t (49)

(i.e., *some* of their moments must be equal), where m is the number of measurements. These conditions arise simply analysing the plausibilities $P(R_i \mid M^k \wedge I)$ and (47) first by means of one set of circumstances, then by means of the other, equating the expressions thus obtained, and applying property (39) (under the assumptions (43) and (46)). \square

Remark 12. The formalism lends naturally itself also to iteration. One can introduce ‘circumstances of circumstances’, etc., i.e. deeper and deeper levels of analysis for the context I . What mathematically comes about looks like a hierarchy of ‘plausibilities of plausibilities’, ‘plausibilities of plausibilities of plausibilities’, etc., which

¹⁸Note that the position of the mass-centre of a coin is not a very important factor in the assignment of a plausibility to heads or tails. See Jaynes’ discussion [20, ch. 10].

Good calls ‘probabilities of Type I, II, III’, etc. [61]. Of course, such a cornucopia of recursive analyses may be appropriate and useful in some cases, while in others may just lead to constipation. \square

Remark 13. The interpretation here presented may also provide another point of view on theories of interval-valued probabilities (see e.g. [123, 124][75, esp. § 3.1] and cf. [61, § 2.2][125]), and in this sense completes or re-interprets studies by e.g. Jami-son [62], Levi [125, 126], Fishburn [127], Nau [128]. \square

5 Conclusions

We often have the need to use statistical models with plausibility-like parameters, especially in classical and quantum mechanics, and must face the problems of choosing an suitable parameter space and a plausibility distribution on this space. These problems would sometimes be less difficult if the parameters could be given some interpretation.

Some interpret the parameters as ‘propensities’ or ‘physical probabilities’. But these concepts do not make sense to us.

De Finettians say that we should not interpret the parameters, but think in terms of infinitely exchangeable sequences instead; the parameters and their priors then arise as mathematical devices. But we do not like being forced to think in terms of infinite sequences, whose vast majority (∞) of elements must then necessarily be fictitious. And there are situations that can be repeated a finite number of times only.

In addition to this, looking at concrete applications of statistical models it seems that behind the parameters we often have ‘at the back of our minds’ an idea of some possible hypotheses — ‘circumstances’ — that could hold in the context under study, e.g. a physical measurement. These circumstances could help us in the assignment of plausibilities. And they need not concern ‘causes’ or ‘propensities’; see remarks 1 and 5. At the same time, we are sometimes not interested in the intrinsic details of such circumstances, but only in the plausibilities that we eventually assign on their grounds.

We have seen in this study that plausibility theory allows us, starting from *any* set $\{C_j\}$ of circumstances, to form another, ‘coarse-grained’ set $\{S_q\}$ with the property that its circumstances lead each one to a different plausibility distribution. The circumstances of this set can then be uniquely indexed by the plausibility distributions they lead us to assign. This set, moreover, is invariant with respect to changes in the plausibilities of the initial and the coarse-grained sets of circumstances, $\{P(C_j|I)\}$ and $\{P(S_q|I)\}$.

This suggests that plausibility-like parameters like q , when used as arguments of plausibility formulae, can always be interpreted to stand for some appropriately indexed circumstances like S_q . With mathematical care, this may even hold for parameters of continuous statistical models. Parameter priors like $f(q|I)$ can consequently be interpreted as plausibilities of circumstances $P(S_q|I)$.

The study of how these priors are updated when repetitions of ‘similar’ measurements occur, and of particular applications to classical and quantum mechanics, are developed in the next two papers.

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